

Criticality and Averaging in Cosmology

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(Received August 13, 1999)

We propose comparing cosmological solutions in terms of their total spatial volumes $V(\tau)$ as functions of proper time τ , assuming synchronous gauge, and with this intention evaluate the variations of $V(\tau)$ about the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions for dust. This can be done successfully in a simple manner without solving perturbation equations. In particular, we find that first variations vanish with respect to all directions which do not possess homogeneity and isotropy preserving components; in other words, every FLRW solution is a *critical point* for $V(\tau)$ in the properly restricted subspace of the space of solutions. This property may support a validity of the interpretation of the FLRW solutions as constituting an averaged model. We also briefly investigate the second variations of $V(\tau)$.

§1. Introduction

The most important practical models in relativistic cosmology are the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) models, which assume the homogeneity and isotropy of spatial geometry and matter distribution. Since our universe seems homogeneous and isotropic over a large scale (typically the size of a supercluster), it may be natural to expect that such a model provides a good approximation for the real universe, describing averaged properties of the geometry and matter distribution in it. However,¹⁾ justification of such an expectation should be considered carefully, since Einstein's equation is highly nonlinear, and thus no simple averaging procedure applied to spatial geometry and matter commutes with time evolution.

The averaging problem in general concerns how one can modify Einstein's equation for averaged variables to be compatible with the time evolution of the original inhomogeneous data for the variables. To analyse the problem one usually has to determine an averaging procedure. In fact, many averaging procedures have been proposed,²⁾⁻¹¹⁾ but the dynamical properties of the resulting averaged variables are often unclear, and little firm agreement is seen among them.

In this paper we discuss the problem in such an indirect way that no explicit averaging procedure is introduced, so that the results do not depend on the averaging procedure. We simply *compare* the dynamical properties of the solutions — through the temporal behavior of their total spatial volume $V(\tau)$, and we find a striking property of the FLRW solution for dust matter. One of the solutions that is compared is an FLRW solution $g_{ab}^{(0)}$, and the other is an inhomogeneous solution g_{ab} . Synchronous and comoving conditions are assumed. The reason we focus on the volume $V(\tau)$ is that the FLRW model has only one dynamical variable, the scale factor $a(\tau)$, and its dynamical content is equivalent to the volume $V_0(\tau)$. Thus, it is

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reasonable to extract $V(\tau)$ also for an inhomogeneous solution and use it to compare the solutions. To be specific, one could suppose the norm of the difference between the solutions defined by, e.g., the L^p -norm applied to the volumes,

$$\|g_{ab} - g_{ab}^{(0)}\|_p = \left(\int_{-\infty}^{\infty} |V(\tau) - V_0(\tau)|^p d\tau \right)^{1/p}, \quad (1.1)$$

though we will not use any specific norm. The difference $\Delta V(\tau) \equiv V(\tau) - V_0(\tau)$ itself is the object on which we focus.

Our main claim is that each element of the two-parameter set of the FLRW solutions for dust is a *critical point* for $V(\tau)$, i.e., the variations of $V(\tau)$ in the space of solutions about the FLRW solution vanish: $\delta V(\tau) = 0$. (This statement shall be made more precise.) Due to this property the temporal behavior of an inhomogeneous solution which is almost homogeneous and isotropic is the same as that of an FLRW solution up to first order in the sense that $\Delta V(\tau) \simeq \delta V(\tau) = 0$. This implies that there exists a natural correspondence between inhomogeneous solutions and the FLRW solution. We interpret this correspondence as an *averaging* (up to first order). On the other hand the second variations $\delta^2 V(\tau)$ do not vanish in general. We will also give some formulae for the second variations.

The content of this paper is basically a direct generalization of a previous paper,¹²⁾ where we examined the spherically symmetric case using the exact Lemaître-Tolman-Bondi solution. The method used to prove the claim in this paper is different from that in the previous paper, since we cannot rely upon an exact solution. The existence of a conserved quantity (the total number of dust particles) will be one of the key points of our analysis. Another key is the property of a homogeneous and isotropic (three-)metric, mainly as an Einstein metric, i.e., that the Ricci tensor is proportional to the metric. These two features give rise to a closed ordinary differential equation for $\delta V(\tau)$, and which is used to show the claim. In particular, we do not have to solve perturbation equations. The second variation $\delta^2 V(\tau)$ is obtained in a similar way.

Our approach is not to present a complete framework for averaging, but rather to give a transparent view of the interrelation between inhomogeneous solutions and the FLRW solutions. This should be a helpful guide for a detailed investigation.

In the next section we make some preparations for our analysis and introduce some notation. In §3 we discuss the first variations of $V(\tau)$ mainly in the case of closed spatial manifolds. Section 4 concerns the second variations. In §5 we comment on cases of spatial manifolds which are compact but have boundaries. Section 6 is devoted to a summary.

§2. Preliminary

Let M be a 3-dimensional compact manifold which admits a constant curvature metric \tilde{h}_{ij} . We use x^i to denote coordinates for M . The spacetime manifold we consider is the direct product $M \times \mathbf{R}$, for which we use $x^0 = \tau$ to denote the

coordinate for \mathbf{R} . We consider smooth metrics on $M \times \mathbf{R}$ of the synchronous form

$$ds^2 = g_{ab}dx^a dx^b = d\tau^2 - h_{ij}(\tau, x^i)dx^i dx^j. \quad (2.1)$$

Here the spatial metric $h_{ij}(\tau, x^i)$ can be considered as a one-parameter (τ) family of smooth metrics on M . Conversely, once given a one-parameter family $h_{ij}(\tau, x^i)$ of smooth metrics on M , we can identify it with the spacetime metric (2.1). Thus, we can define the space P^* of smooth (synchronous) metrics on $M \times \mathbf{R}$ as the set of all possible one-parameter families of smooth metrics on M , $h_{ij}(\tau, x^i)$. Next, let us consider Einstein's equation for dust: $G_{ab} = 8\pi\rho u_a u_b$, where G_{ab} is the Einstein tensor for g_{ab} , and ρ and $u^a \equiv (\partial/\partial\tau)^a$ are, respectively, the matter density and four-velocity of dust. We define the space of solutions $P \subset P^*$ as the set of all smooth solutions of this equation. We distinguish P and P^* to clarify whether a relation holds with dynamical equations or without them.

Since we have assumed that M admits a constant curvature metric \tilde{h}_{ij} , the spacetime manifold admits a spatially homogeneous and isotropic metric of the form

$$ds^2 = g_{ab}^{(0)}dx^a dx^b = d\tau^2 - a(\tau)^2 \tilde{h}_{ij}(x^i)dx^i dx^j. \quad (2.2)$$

The function $a(\tau)$ specifies the scale of M for a slice $\tau = \text{constant}$, and it is called the *scale factor*. To make the magnitude of a well-defined, we remove the scale ambiguity of \tilde{h}_{ij} by choosing it as a standard metric, satisfying

$${}^{(3)}\tilde{R} = 6k, \quad (2.3)$$

where ${}^{(3)}\tilde{R}$ is the scalar curvature for the standard metric chosen, and $k = 1$ for a positive curvature space, $k = -1$ for a negative curvature space, and $k = 0$ for a flat space. The constant k is referred to as the *curvature index*. The curvature tensor ${}^{(3)}R_{ijkl}$, Ricci tensor ${}^{(3)}R_{ij}$, and scalar curvature ${}^{(3)}R$ for the spatial metric $h_{ij} = a^2 \tilde{h}_{ij}$ can be found, for example, from the conformal invariance of the Ricci tensor ${}^{(3)}R_{ij}$:

$${}^{(3)}R_{ij} = {}^{(3)}\tilde{R}_{ij} = 2k \tilde{h}_{ij} = \frac{2k}{a^2} h_{ij}. \quad (2.4)$$

The curvature tensor and scalar curvature are hence

$${}^{(3)}R_{ijkl} = \frac{k}{a^2} (h_{ik} h_{jl} - h_{il} h_{jk}), \quad {}^{(3)}R = \frac{6k}{a^2}. \quad (2.5)$$

We define the total spatial volume $V(\tau)$ for the metric (2.1) by

$$V(\tau) \equiv \int_M \sqrt{h} d^3x, \quad (2.6)$$

and the total particle number E by

$$E \equiv \int_M G_{00} \sqrt{h} d^3x = \frac{1}{2} \int_M ({}^{(3)}R + K^2 - K_{ij} K^{ij}) \sqrt{h} d^3x, \quad (2.7)$$

where ${}^{(3)}R$ is the scalar curvature for the spatial metric h_{ij} , $K_{ij} = \frac{1}{2} \dot{h}_{ij}$ is the extrinsic curvature, and $K \equiv h^{ij} K_{ij}$, $h \equiv \det(h_{ij})$. To raise and lower the spatial indices

i, j, \dots we use h^{ij} and h_{ij} , e.g., $K^i_j \equiv h^{ik}K_{kj}$. Here dots (\cdot) represent derivatives with respect to τ .

The particle number E is conserved if the dynamical and constraint equations $G_{ij} = 0 = G_{0i}$ are imposed. In fact, one can check by a straightforward calculation that $(G_{00}\sqrt{h})' = (-G_{ij}K^{ij} + \nabla^i G_{0i})\sqrt{h}$, implying $\dot{E} = 0$ if we impose the equations.

$V(\tau)$ can be regarded as a “function-valued” functional on P (or P^*), since once $h_{ij}(\tau, x^i) \in P$ (or P^*) is specified the function $V(\tau)$ is determined. (To express this we could add an argument of the functional as “ $V(\tau)[h_{ij}(\tau)]$,” but for simplicity we shall write “ $V(\tau)$.”) Similarly, E is also a functional on P or P^* , which is real-valued on P , since E is conserved, but is function-valued on P^* .

Consider a smooth path l in P^* , $l : [0, \infty) \rightarrow P^*$. Let $V_\epsilon(\tau)$ be the function $V(\tau)$ for $l(\epsilon)$, where $\epsilon \in [0, \infty)$. Then, we can expand $V_\epsilon(\tau)$ at $\epsilon = 0$ as

$$V_\epsilon(\tau) = V_0(\tau) + \epsilon \delta V(\tau) + \frac{1}{2} \epsilon^2 \delta^2 V(\tau) + \dots, \quad (2.8)$$

where

$$\delta V(\tau) \equiv \left. \frac{dV_\epsilon(\tau)}{d\epsilon} \right|_{\epsilon=0}, \quad \delta^2 V(\tau) \equiv \left. \frac{d^2 V_\epsilon(\tau)}{d\epsilon^2} \right|_{\epsilon=0}, \quad \dots \quad (2.9)$$

We call $\delta V(\tau)$ and $\delta^2 V(\tau)$ the *variation* and *second variation* of $V(\tau)$, respectively. Variations of any functional in any space are defined similarly.

Practically we will not specify any path, but we expand a variation of a functional in terms of variations of h_{ij} and K_{ij} . Thus it is useful to introduce some notation for them.

Notation Let δh_{ij} and δK_{ij} be variations of the spatial metric and extrinsic curvature for the spacetime metric (2.1). Similarly, let $\delta^2 h_{ij}$ and $\delta^2 K_{ij}$ be their second variations. We use the following notation for them and their traces throughout this paper:

$$\gamma_{ij} \equiv \delta h_{ij}, \quad \gamma \equiv h^{ij} \gamma_{ij}, \quad \lambda_{ij} \equiv \delta K_{ij}, \quad \lambda \equiv h^{ij} \lambda_{ij}, \quad (2.10)$$

$$\gamma^{(2)}_{ij} \equiv \delta^2 h_{ij}, \quad \gamma^{(2)} \equiv h^{ij} \gamma^{(2)}_{ij}, \quad \lambda^{(2)}_{ij} \equiv \delta^2 K_{ij}, \quad \lambda^{(2)} \equiv h^{ij} \lambda^{(2)}_{ij}. \quad (2.11)$$

§3. Variations of V

Our concern is the variations of $V(\tau)$ and E evaluated about the homogeneous and isotropic metric (2.2). We find a closed relation between them if the spatial manifold M is closed, but for later convenience we present the relation in the case that M is compact first.

Lemma 1 *Let $V(\tau)$ and E be the functionals on P^* defined by Eqs.(2.6) and (2.7). For the variations about a spatially homogeneous and isotropic metric $g_{ab}^{(0)}$,*

$$\delta E - a \delta B = \left(\frac{k}{a^2} - 3 \left(\frac{\dot{a}}{a} \right)^2 \right) \delta V + 2 \frac{\dot{a}}{a} \delta \dot{V}, \quad (3.1)$$

where

$$\delta B \equiv \frac{1}{2a} \int_M \nabla^i (\nabla^j \gamma_{ij} - \nabla_i \gamma) \sqrt{h} d^3 x, \quad (3.2)$$

$a = a(\tau)$ is the scale factor for the homogeneous and isotropic spatial metric, and k is the curvature index.

Here, $\delta\dot{V}$ is the time derivative of δV (“ δ ” and “ \cdot ” commute, however), and ∇_i represents the covariant derivative associated with (the zeroth order of) h_{ij} . As long as there is no confusion, we omit superscripts (such as that on $h_{ij}^{(0)}$) in reference to a zeroth order quantity.

To prove Lemma 1 we present some formulae, which can all be checked by straightforward calculations.

Formulae 2 The variations δV , $\delta\dot{V}$, and δE (about a generic point) are given by

$$\delta V = \frac{1}{2} \int_M \gamma \sqrt{h} d^3x, \quad (3.3)$$

$$\delta\dot{V} = \int_M \left(\lambda + K\gamma - K_{ij}\gamma^{ij} \right) \sqrt{h} d^3x \quad (3.4)$$

and

$$\begin{aligned} \delta E = \int_M \left[- \left(\frac{1}{2} {}^{(3)}R^{ij} + K K^{ij} - K^i{}_k K^{jk} \right) \gamma_{ij} \right. \\ \left. + \frac{1}{4} \left({}^{(3)}R + K^2 - K_{ij} K^{ij} \right) \gamma - \left(K^{ij} - K h^{ij} \right) \lambda_{ij} \right. \\ \left. + \frac{1}{2} \nabla^i (\nabla^j \gamma_{ij} - \nabla_i \gamma) \right] \sqrt{h} d^3x. \end{aligned} \quad (3.5)$$

Proof of Lemma 1. We evaluate the variations (3.3), (3.4) and (3.5) about a homogeneous and isotropic metric. The spatial part of such a metric can be written as $h_{ij} = a^2(\tau) \tilde{h}_{ij}$, as in Eq.(2.2). The extrinsic curvature becomes

$$K_{ij} = \frac{1}{2} \dot{h}_{ij} = a \dot{a} \tilde{h}_{ij} = \frac{\dot{a}}{a} h_{ij}. \quad (3.6)$$

The Ricci tensor ${}^{(3)}R_{ij}$ is also proportional to h_{ij} (Eq.(2.4)). If we substitute Eqs.(3.6) and (2.4) into Eqs.(3.4) and (3.5), we find

$$\delta\dot{V} = \int_M \left(\lambda + \frac{1}{2} \frac{\dot{a}}{a} \gamma \right) \sqrt{h} d^3x = \int_M \lambda \sqrt{h} d^3x + \frac{\dot{a}}{a} \delta V \quad (3.7)$$

and

$$\delta E - a \delta B = \frac{1}{2} \left(\frac{k}{a^2} - \left(\frac{\dot{a}}{a} \right)^2 \right) \int_M \gamma \sqrt{h} d^3x + 2 \frac{\dot{a}}{a} \int_M \lambda \sqrt{h} d^3x, \quad (3.8)$$

where δB is defined by Eq.(3.2). The last equation (3.8) implies Eq.(3.1) if we note Eqs.(3.3) and (3.7). \square

If the spatial manifold M is *closed* (i.e., $\partial M = \emptyset$), the divergence term δB vanishes, so that we have a simpler result:

Corollary 3 Let $V(\tau)$ and E be the functionals on P^* defined by Eqs.(2.6) and (2.7). If the spatial manifold M is closed, for variations about a spatially homogeneous and isotropic metric $g_{ab}^{(0)}$,

$$\delta E = \left(\frac{k}{a^2} - 3 \left(\frac{\dot{a}}{a} \right)^2 \right) \delta V + 2 \frac{\dot{a}}{a} \delta\dot{V}, \quad (3.9)$$

where $a = a(\tau)$ is the scale factor for the homogeneous and isotropic spatial metric, and k is the curvature index.

The relation (3.9) also holds for the variations taken in P . In addition, we can regard δE as a constant, since E is conserved in such a case. The scale factor $a(\tau)$ is also a definite function which obeys the Einstein equation

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = 0. \quad (3.10)$$

Hence the relation (3.9) is a closed linear ordinary differential equation for δV . Although we can easily find the solution for Eq.(3.9) directly, for later convenience we present a more general formula first:

Formula 4 *The general solution for the ordinary differential equation for $f = f(\tau)$*

$$g(\tau) = \left(\frac{k}{a^2} - 3\left(\frac{\dot{a}}{a}\right)^2\right) f + 2\frac{\dot{a}}{a} \dot{f}, \quad (3.11)$$

where $g(\tau)$ is a given function of τ and $a = a(\tau)$ is a solution for Eq.(3.10), is given by

$$f(\tau) = a^2 \dot{a} \left(\frac{1}{2} \int \frac{g(\tau)}{a \dot{a}^2} d\tau + c \right), \quad (3.12)$$

where c is an integration constant.

Applying this formula we at once obtain the following result.

Lemma 5 *Let $V(\tau)$ be the functional in the space of solutions P defined by Eq.(2.6). If the spatial manifold M is closed, variations about a homogeneous and isotropic solution are given by*

$$\delta V(\tau) = a^2 \dot{a} \left(\frac{\delta E}{2} \int \frac{d\tau}{a \dot{a}^2} + \delta C \right), \quad (3.13)$$

where δE and δC are constants, and $a = a(\tau)$ is a solution of Eq.(3.10).

We present the explicit forms of the solution for completeness:

(i) $k = 1$: $a(\tau) = a_0(1 - \cos \eta)$, $\tau - \tau_{0c} = a_0(\eta - \sin \eta)$,

$$\delta V(\tau) = a_0^2(1 - \cos \eta) \left(\mathcal{A}(\eta) \frac{\delta E}{2} + \sin \eta \delta C \right), \quad (3.14)$$

where $\mathcal{A}(\eta) \equiv (1 - \cos \eta)^2 - \sin \eta(\eta - \sin \eta) = 2(1 - \cos \eta) - \eta \sin \eta$.

(ii) $k = -1$: $a(\tau) = a_0(\cosh \eta - 1)$, $\tau - \tau_{0c} = a_0(\sinh \eta - \eta)$,

$$\delta V(\tau) = a_0^2(\cosh \eta - 1) \left(\mathcal{A}_-(\eta) \frac{\delta E}{2} + \sinh \eta \delta C \right), \quad (3.15)$$

where $\mathcal{A}_-(\eta) \equiv (\cosh \eta - 1)^2 - \sinh \eta(\sinh \eta - \eta) = \eta \sinh \eta - 2(\cosh \eta - 1)$.

(iii) $k = 0$: $a(\tau) = a_0(\tau - \tau_{0c})^{2/3}$,

$$\delta V(\tau) = \frac{3}{4}(\tau - \tau_{0c})^2 \delta E + \frac{2}{3}a_0^3(\tau - \tau_{0c}) \delta C. \quad (3.16)$$

In these solutions a_0 and τ_{0c} are constant parameters. As one can easily see, τ_{0c} is redundant as far as the FLRW solutions are concerned, since it carries the gauge

freedom associated with the choice of the origin of time. However, we will see that τ_{0c} is of some importance in the wider context in which we are interested.

Does the constant δC in Eq.(3.13) defined as an integration constant for the differential equation (3.9) have an “integral” C such that its variation coincides with the δC that connects variations of E and $V(\tau)$ through Eq.(3.13)? We may expect that at least in a neighborhood of the FLRW solutions there exist such a single-valued functional C . We will call this functional C , a *big bang constant*. For example, in the space of spherically symmetric solutions there exists a natural choice of C that is globally defined,¹²⁾ but in the general case, global existence of C , of course, requires proof. However, we do not discuss this problem further, since local existence is sufficient for our purposes. We do not discuss the uniqueness of C either, for the same reason. (In fact, it would not be unique from the definition above. A complete definition may be given after higher order variations of $V(\tau)$ are specified.) We will see that considering the functional C is convenient for our analysis.

The function $V(\tau)$ for an FLRW solution, denoted $V_0(\tau)$ hereafter, depends parametrically on a_0 and τ_{0c} , and thus it varies if a_0 and τ_{0c} are varied (with τ fixed). For example, in the case $k = 1$ with $M \simeq S^3$, the total derivative dV_0 with respect to a_0 and τ_{0c} is¹²⁾

$$dV_0(\tau) = 6\pi^2 a_0^2 (1 - \cos \eta) (\mathcal{A}(\eta) da_0 - \sin \eta d\tau_{0c}). \quad (3.17)$$

Note that the time dependence of this function is the same as that of Eq.(3.14). Similar results hold for the other cases of k . To understand the implication of this fact, we first need to give some definitions.

Definitions Let O be the two-dimensional space of the FLRW solutions, and let $o(a_0, \tau_{0c}) \in O$ be the FLRW solution with given FLRW parameters a_0 and τ_{0c} . O is a subspace of P , as well. Let $T_{a_0, \tau_{0c}}(P)$ be the tangent space at $o(a_0, \tau_{0c}) \in P$. The tangent space at $o(a_0, \tau_{0c}) \in O$ can be considered as a subspace of $T_{a_0, \tau_{0c}}(P)$, and we denote it $O_{a_0, \tau_{0c}}(P)$. We call $O_{a_0, \tau_{0c}}(P)$ the *tangent space which preserves the homogeneity and isotropy*.

In correspondence to $O_{a_0, \tau_{0c}}(P)$ we can define a complement $Q_{a_0, \tau_{0c}}(P)$ by $T_{a_0, \tau_{0c}}(P) = O_{a_0, \tau_{0c}}(P) \oplus Q_{a_0, \tau_{0c}}(P)$, where \oplus is the direct sum. $Q_{a_0, \tau_{0c}}(P)$ is not unique at this point, but a comparison of Eqs.(3.14) and (3.17) reveals that a tangent space at $o(a_0, \tau_{0c})$ spanned by independent tangent vectors with nonvanishing δE and δC is naturally identified with the tangent space $O_{a_0, \tau_{0c}}(P)$. This identification makes $Q_{a_0, \tau_{0c}}(P)$ unique, since we can require that for any $\frac{\delta}{\delta h_{ij}} \in Q_{a_0, \tau_{0c}}(P)$ the corresponding variation of $V(\tau)$ vanish, $\delta V(\tau) = 0$. To summarize, we have:

Theorem 6 *Let $V(\tau)$ be the functional in the space P of solutions defined by Eq.(2.6), and suppose that the spatial manifold M is closed. Then, the tangent space $T_{a_0, \tau_{0c}}(P)$ at $o(a_0, \tau_{0c}) \in P$ can be uniquely decomposed as the direct sum*

$$T_{a_0, \tau_{0c}}(P) = O_{a_0, \tau_{0c}}(P) \oplus Q_{a_0, \tau_{0c}}(P), \quad (3.18)$$

where $Q_{a_0, \tau_{0c}}(P)$ is the space for which the action of any $\frac{\delta}{\delta h_{ij}} \in Q_{a_0, \tau_{0c}}(P)$ on $V(\tau)$ vanishes:

$$\delta V(\tau) = 0, \quad (3.19)$$

and $O_{a_0, \tau_{0c}}(P)$ is the tangent space which preserves the homogeneity and isotropy.

In the sense of Eq.(3.19) an FLRW solution is a *critical point* for $V(\tau)$ in the space of solutions. We may interpret this fact in connection with the averaging as follows. Let $p \in P$ be an inhomogeneous solution which is almost homogeneous and isotropic, and let $V_p(\tau)$ be its volume as a function of time. Then, the theorem implies that there exist the set of values (a_0, τ_{0c}) for which the difference between the volume $V_0(\tau)$ of the FLRW solution $o(a_0, \tau_{0c})$ and that of the inhomogeneous solution is the same up to first order: $\Delta V(\tau) \equiv V_p(\tau) - V_0(\tau) \simeq \delta V(\tau) = 0$.

This set of values (a_0, τ_{0c}) is determined by the conditions $\delta E = 0$ and $\delta C = 0$. In other words, the best fit FLRW solution $o(a_0, \tau_{0c})$ is the solution in O that possesses the same particle number E and the same big bang constant C as those of the solution p . This correspondence $p \rightarrow o(a_0, \tau_{0c})$ defines the map $Av : P \rightarrow O$. That is:

Definitions The space of solutions P is foliated by the level sets for E (i.e., the sets $E = \text{constant}$), and it is also foliated by the level sets for C . Let $P_{E,C} \subset P$ be the double level set $E = \text{constant}$ ($= E$) and $C = \text{constant}$ ($= C$). In $P_{E,C}$ there exists only one FLRW solution $o(a_0, \tau_{0c})$. The map Av is defined by this correspondence:

$$Av : P_{E,C} \rightarrow o(a_0, \tau_{0c}). \quad (3.20)$$

We call this map the *averaging map* in the space of solutions.

Strictly speaking, the map Av can be defined only on the domain where the big bang constant C is defined. This domain exists at least in a neighborhood of O . Again, this is sufficient for our purposes.

In terms of the averaging map Av , we can reword Theorem 6 as follows.

Theorem 7 *The variation of $V(\tau)$ along any smooth path in $Av^{-1}o(a_0, \tau_{0c})$ from an FLRW solution $o(a_0, \tau_{0c})$ vanishes:*

$$\delta V(\tau) = 0. \quad (3.21)$$

§4. Second variations of V

The variation of $V(\tau)$ about an FLRW solution is essentially zero in the sense of Theorem 6 (or 7). The second variation $\delta^2 V(\tau)$ therefore gives the leading contribution to the difference $\Delta V(\tau)$. In this section we evaluate $\delta^2 V(\tau)$ in a manner similar to the previous section.

We first obtain the following.

Lemma 8 *Let $V(\tau)$ and E be the functionals on P^* defined by Eqs.(2.6) and (2.7), and suppose that the spatial manifold M is closed. For the second variations about a spatially homogeneous and isotropic metric $g_{ab}^{(0)}$,*

$$\delta^2 E = \left(\frac{k}{a^2} - 3 \left(\frac{\dot{a}}{a} \right)^2 \right) \delta^2 V + 2 \frac{\dot{a}}{a} \delta^2 \dot{V} + \Gamma(\tau), \quad (4.1)$$

where $\Gamma(\tau)$ is the function of τ that is determined from the first variation of the

metric, defined by

$$\begin{aligned} \Gamma(\tau) \equiv \int_M \bigg[& \left(\frac{\dot{a}}{a}\right)^2 \gamma^2 - \frac{1}{2} \left(\frac{k}{a^2} + 2 \left(\frac{\dot{a}}{a}\right)^2\right) \gamma^{ij} \gamma_{ij} \\ & + \frac{1}{4} \left((\nabla_i \gamma)(\nabla^i \gamma) - (\nabla_i \gamma_{jk})(\nabla^i \gamma^{jk}) \right) \\ & + \frac{1}{2} \left((\nabla_i \gamma^i_k)(\nabla_j \gamma^{jk}) - (\nabla_i \gamma)(\nabla_j \gamma^{ij}) \right) \\ & + 2 \frac{\dot{a}}{a} \left(\gamma^{ij} \lambda_{ij} - \gamma \lambda \right) + \lambda^2 - \lambda^{ij} \lambda_{ij} \bigg] \sqrt{h} d^3x. \end{aligned} \quad (4.2)$$

Here, $a = a(\tau)$ is the scale factor in $g_{ab}^{(0)}$, and k is the curvature index.

To prove Lemma 8 we need to calculate the second variations $\delta^2 E$ and $\delta^2 V(\tau)$, and the time derivative $\delta^2 \dot{V}$, about a generic point in P^* or P . We obtain:

Formulae 9 The second variations $\delta^2 V$, $\delta^2 \dot{V}$ and $\delta^2 E$ are given by

$$\delta^2 V = \frac{1}{2} \int_M \left(\gamma^{(2)} - \gamma^{ij} \gamma_{ij} + \frac{1}{2} \gamma^2 \right) \sqrt{h} d^3x, \quad (4.3)$$

$$\begin{aligned} \delta^2 \dot{V} = \int_M \bigg[& \lambda^{(2)} - 2\gamma^{ij} \lambda_{ij} + \gamma \lambda - K^{ij} \left(\gamma^{(2)}_{ij} - 2\gamma_{ik} \gamma_j^k + \gamma \gamma_{ij} \right) \\ & + \frac{1}{2} K \left(\gamma^{(2)} - \gamma^{ij} \gamma_{ij} + \frac{1}{2} \gamma^2 \right) \bigg] \sqrt{h} d^3x, \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \delta^2 E = \int_M \bigg[& \frac{1}{2} {}^{(3)}R_{ikjl} \gamma^{ij} \gamma^{kl} + \frac{1}{2} {}^{(3)}R_{ij} \left(-\gamma^{(2)ij} + \gamma^i_k \gamma^{jk} - \gamma \gamma^{ij} \right) \\ & - \frac{1}{4} {}^{(3)}R \left(-\gamma^{(2)} + \gamma^{ij} \gamma_{ij} - \frac{1}{2} \gamma^2 \right) \\ & + \frac{1}{4} \gamma^{ij} \left(\nabla^k \nabla_k \gamma_{ij} + \nabla_i \nabla_j \gamma - 2 \nabla_i \nabla_k \gamma_j^k \right) + \frac{1}{4} \gamma \nabla^i \left(\nabla^j \gamma_{ij} - \nabla_i \gamma \right) \\ & + \frac{1}{2} \nabla_i (\delta \beta^i + \frac{1}{2} \gamma \beta^i) \\ & + \left(K^{ij} K^{kl} - K^{ik} K^{jl} \right) \gamma_{ij} \gamma_{kl} \\ & + \left(K_i^k K_{jk} - K K_{ij} \right) \left(\gamma^{(2)ij} - 2\gamma^{il} \gamma_j^k + \gamma \gamma^{ij} \right) \\ & + \frac{1}{4} \left(K^2 - K^{ij} K_{ij} \right) \left(\gamma^{(2)} - \gamma^{ij} \gamma_{ij} + \frac{1}{2} \gamma^2 \right) \\ & - K^{ij} \left(\lambda^{(2)}_{ij} - 4\gamma_{ik} \lambda_j^k + 2\gamma_{ij} \lambda + \gamma \lambda_{ij} \right) \\ & + K \left(\lambda^{(2)} - 2\gamma^{ij} \lambda_{ij} + \gamma \lambda \right) - \lambda^{ij} \lambda_{ij} + \lambda^2 \bigg] \sqrt{h} d^3x, \end{aligned} \quad (4.5)$$

where

$$\beta_i \equiv \nabla^j \gamma_{ij} - \nabla_i \gamma \quad (4.6)$$

and $\delta\beta^i$ is the variation of $\beta^i = h^{ij}\beta_j$.

Proof of Lemma 8. We evaluate Eqs.(4.4) and (4.5) about the FLRW solution, using Eqs.(2.4), (2.5), and (3.6). The result for $\delta^2\dot{V}$ is

$$\begin{aligned}\delta^2\dot{V} &= \int_M \left(\lambda^{(2)} - 2\gamma^{ij}\lambda_{ij} + \gamma\lambda \right) \sqrt{h}d^3x + \frac{\dot{a}}{2a} \int_M \left(\gamma^{(2)} + \gamma^{ij}\gamma_{ij} - \frac{1}{2}\gamma^2 \right) \sqrt{h}d^3x \\ &= \int_M \left(\lambda^{(2)} - 2\gamma^{ij}\lambda_{ij} + \gamma\lambda \right) \sqrt{h}d^3x + \frac{\dot{a}}{a} \delta^2V \\ &\quad + \frac{\dot{a}}{a} \int_M \left(\gamma^{ij}\gamma_{ij} - \frac{1}{2}\gamma^2 \right) \sqrt{h}d^3x,\end{aligned}\tag{4.7}$$

where in the last equality we have used Eq.(4.3). A straightforward evaluation of δ^2E gives

$$\begin{aligned}\delta^2E &= \frac{1}{2} \left(\frac{k}{a^2} - \left(\frac{\dot{a}}{a} \right)^2 \right) \int_M \gamma^{(2)} \sqrt{h}d^3x + 2\frac{\dot{a}}{a} \int_M \lambda^{(2)} \sqrt{h}d^3x \\ &\quad + \int_M \left[\frac{1}{4} \left(\frac{k}{a^2} - \left(\frac{\dot{a}}{a} \right)^2 \right) \gamma^2 - \left(\frac{k}{a^2} + \frac{3}{2} \left(\frac{\dot{a}}{a} \right)^2 \right) \gamma^{ij}\gamma_{ij} \right. \\ &\quad + \frac{1}{4} \gamma^{ij} (\nabla^k \nabla_k \gamma_{ij} + \nabla_i \nabla_j \gamma - 2\nabla_i \nabla_k \gamma_j^k) + \frac{1}{4} \gamma \nabla^i (\nabla^j \gamma_{ij} - \nabla_i \gamma) \\ &\quad \left. - 2\frac{\dot{a}}{a} \gamma^{ij}\lambda_{ij} + (\lambda^2 - \lambda^{ij}\lambda_{ij}) \right] \sqrt{h}d^3x.\end{aligned}\tag{4.8}$$

We have dropped the divergence term $\frac{1}{2} \int \nabla_i (\delta\beta^i + \frac{1}{2}\gamma\beta^i) \sqrt{h}d^3x$. Finally, applying integration by parts to the terms containing covariant derivatives, and using Eqs.(4.3) and (4.7), we obtain Eq.(4.1) with Eq.(4.2). \square

If the variations are taken in P , δ^2E is a constant. The function $\Gamma(\tau)$ is determined from the first variation of the metric, and thus it can be thought of as a given function. Applying Formula 4, we obtain

Lemma 10 *Let $V(\tau)$ be the functional in the space of solutions P defined by Eq.(2.6), and suppose that the spatial manifold M is closed. Then, the second variation about a homogeneous and isotropic solution is given by*

$$\delta^2V(\tau) = a^2\dot{a} \left(\frac{\delta^2E}{2} \int \frac{d\tau}{a\dot{a}^2} + \delta^2C \right) - \frac{a^2\dot{a}}{2} \int \frac{\Gamma(\tau)}{a\dot{a}^2} d\tau,\tag{4.9}$$

where δ^2E and δ^2C are constants, $a = a(\tau)$ is a solution of Eq.(3.10), and $\Gamma(\tau)$ is defined by Eq.(4.2).

The integration constant δ^2C appearing above is naturally identified with the variation of δC appearing in Lemma 5, since their coefficients are the same. The constant δ^2C is therefore the second variation of the big bang constant C . Our conclusion in this section is the following (cf. Theorem 7):

Theorem 11 *The second variation of $V(\tau)$ along a smooth path in $Av^{-1}o(a_0, \tau_{0c})$ from an FLRW solution $o(a_0, \tau_{0c})$ is given by*

$$\delta^2V(\tau) = -\frac{a^2\dot{a}}{2} \int \frac{\Gamma(\tau)}{a\dot{a}^2} d\tau,\tag{4.10}$$

where $a = a(\tau)$ is a solution of Eq.(3.10), and $\Gamma(\tau)$ is defined by Eq.(4.2).

§5. Comment on the boundary effect

In the previous two sections we have mainly considered the cases of spatial manifolds being closed, i.e. $\partial M = \emptyset$. Such a case is clearest from a theoretical point of view, since we do not need information on boundaries. However, we may also be interested in a compact spatial manifold with boundaries, for example, a spatial manifold inside the horizon. If we restrict to the case in which the boundaries are comoving so that the particle number E is conserved, we can apply a procedure similar to the ones used in the previous sections. See Lemma 1. Now δB does not vanish, but it can be shown that it is *constant*. In fact, the linearized momentum constraint equation proves $\delta \dot{B} = 0$. Thus we can again integrate Eq.(3.1) using Formula 4, and we obtain

$$\delta V(\tau) = a^2 \dot{a} \left(\frac{\delta E}{2} \int \frac{d\tau}{a \dot{a}^2} - \frac{\delta B}{2} \int \frac{d\tau}{\dot{a}^2} + \delta C \right), \quad (5.1)$$

instead of Eq.(3.13), where δE , δB , and δC are constants.

The explicit forms of the term for the boundary effect

$$\delta V_B(\tau) \equiv -\frac{\delta B}{2} a^2 \dot{a} \int \frac{d\tau}{\dot{a}^2} \quad (5.2)$$

are the following.

(i) For $k = 1$,

$$\delta V_B(\tau) = -\frac{\delta B}{2} a_0^3 (1 - \cos \eta) (4(1 - \cos \eta) + \sin \eta (\sin \eta - 3\eta)). \quad (5.3)$$

(ii) For $k = -1$,

$$\delta V_B(\tau) = -\frac{\delta B}{2} a_0^3 (\cosh \eta - 1) (4(\cosh \eta - 1) - \sinh \eta (3\eta - \sinh \eta)). \quad (5.4)$$

(iii) For $k = 0$,

$$\delta V_B(\tau) = -\frac{\delta B}{2} \frac{9}{10} a_0 (\tau - \tau_{0c})^{\frac{8}{3}}. \quad (5.5)$$

The mode corresponding to $\delta V_B(\tau)$ is not obtained from the space of FLRW solutions.

§6. Summary

We have succeeded in obtaining the variations $\delta V(\tau)$ of the spatial volume about an FLRW solution in the space of solutions for a dust system (Lemma 5). It is noteworthy that this has been done without solving linearized Einstein equations, and the result contains all the cases of curvature index k . Also, we remark that we did *not* split apart $\gamma_{ij} = \delta h_{ij}$ into tensor, vector, and scalar parts, as is done in usual

cosmological perturbation theory (e.g. Ref. 13)). Hence our result does not depend on such a part.

It is, however, more surprising that only two modes (patterns of time-dependence) appear in $\delta V(\tau)$, in spite of the fact that the space of solutions is infinite dimensional. This implies that almost all possible modes vanish. The origin of the two modes is that the FLRW solution is a two-parameter solution. In fact, explicit calculations (see Eq.(3.17)) reveal that the two modes caused by variations with respect to the FLRW parameters (a_0, τ_{0c}) coincide with those found in the (full) space of solutions. Clearly, these modes are not of essential significance in themselves. Instead, their existence tells us that we should divide the space of solutions in such a way that in each piece there exists only one FLRW solution. As we have seen, this can be done by considering the foliation defined by the level sets $E = \text{const}$ and $C = \text{const}$. If the variations are taken in such a set (leaf) (i.e., if $\delta E = 0$ and $\delta C = 0$), the variation of volume $\delta V(\tau)$ completely vanishes (Theorem 6 and 7). Each FLRW solution is therefore a critical point in the leaf that contains only one FLRW solution. Every inhomogeneous solution in such a leaf is therefore naturally related to the unique FLRW solution (cf. Eq.(3.20)), and this correspondence has defined our averaging (up to first order).

The second variations do not vanish in general (See Eq.(4.10)). We cannot therefore define an averaging at second order in the space of solutions. One may need to generalize the space of solutions to define it. This remains as a future work.

Acknowledgments

The author wishes to thank Professor H. Kodama for helpful discussions. He also acknowledges financial support from the Japan Society for the Promotion of Science and the Ministry of Education, Science and Culture.

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